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Common fixed points of hybrid maps and an application

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ABSTRACT

We introduce a new notion that is a generalization of Definition 2.1, Kamran and Cakić (2008) [3]. Using this notion, we establish a new result, that is, coincidence and fixed points for two hybrid pairs of nonself-maps satisfying an implicit relation. This result generalizes the multivalued version of some known results (see, Imdad and Ali (2007) [12] and the references therein). Also, the same result generalizes Theorem 2.8, Liu et al. (2005) [4]. As application, we prove a coincidence point theorem for hybrid nonself-maps in product spaces.

1. Introduction and preliminaries

Since the past five decades, the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. Also, the fixed point theory is a beautiful mixture of analysis, topology, and geometry.

Hybrid contraction maps are contractive conditions involving multivalued mappings and single-valued mappings. Hybrid fixed point theory for these mappings is a new development in the domain of contraction-type multivalued theory (see, e.g., [1–7] and the references therein). Second, several authors proved some common fixed point theorems for nonself-mappings (see, for example, [8–11]). Third, some fixed point theorems for mappings satisfying implicit relations have appeared (see, for instance, [12,13]).

Let X be a metric space with metric d. Then, for $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. Let $CB(X)$ denote the class of all nonempty closed bounded subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X. Let H be the generalized Hausdorff metric on $CL(X)$ generated by the metric d, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\},$$

for every $A, B \in CL(X)$. A point $p \in X$ is said to be a *fixed point* of $I : X \to X$ (resp. $S : X \to CL(X)$) if $p = Ip$ (resp. $p \in Sp$). The point $p \in X$ is said to be a *common fixed point* of $I : X \to X$ and $J : X \to X$ (resp. $I : X \to X$ and $S : X \to CL(X)$) if $p = Ip = Jp$ (resp. $p = Ip \in Sp$). $p \in X$ is called a *coincidence point* of $I : X \to X$ and $J : X \to X$ (resp. $I : X \to X$ and $S : X \to CL(X)$) if $Ip = Jp$ (resp. $Ip \in Sp$). It is obvious that any common fixed point is a coincidence point but its converse need not be true.

Definition 1.1 (*[14]*)**.** Maps $I : X \to X$ and $S : X \to CB(X)$ are *weakly compatible* if they commute at their coincidence points, i.e., $ISx = SIx$ whenever $Ix \in Sx$ for some $x \in X$.

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Definition 1.2 (*[1]*). Let $S : X \to CB(X)$. The map $I : X \to X$ is said to be S-*weakly commuting at* $x \in X$ if $IIx \in SIx$.

The weak compatibility leads to the S-weak commutativity at the coincidence point of I and S but its converse need not be true (see, [1]).

Definition 1.3 (*[15]*). Maps $I : X \to X$ and $J : X \to X$ are said to satisfy *property* (E. A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jx_n = t$ for some $t \in X$.

Definition 1.4 (*[4]*). Let $I, J : X \to X$ and $S, T : X \to CB(X)$. The pairs (I, S) and (J, T) are said to satisfy *the common property* (E. A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X, some $t \in X$, and $A, B \in CB(X)$ such that $\lim_{n\to\infty} Sx_n = A$, $\lim_{n\to\infty} Ty_n = B$ and $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A \cap B$.

Following [13], let Ψ be the family of real lower semi-continuous functions $F : [0, \infty)^6 \to \Re$, $\Re :=$ the set of all real numbers, satisfying the following conditions:

(ψ_1) F is non-increasing in the 3rd, 4th, 5th and 6th coordinate variables,
(ψ_2) there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with
(ψ_{21}) $F(u, v, v, u, u + v, 0) \leq 0$ or (ψ_{22})$F(u, v, u, v, 0, u + v) \leq 0$, we have $u \leq hv$, and
(ψ_3) $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

For the sake of completeness, we enlist some examples which are essentially included in [12,13].

Example 1.5. Define $F : [0, \infty)^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - h\left[a \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\} + (1 - a)\left[\max\left\{t_2^2, t_3t_4, t_5t_6, \frac{1}{2}t_3t_6, \frac{1}{2}t_4t_5\right\}\right]^{\frac{1}{2}}\right],$$

where $h \in (0, 1)$ and $0 \leq a \leq 1$. One can verify that $F \in \Psi$.

Example 1.6. Define $F : [0, \infty)^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - h\left[\max\left\{t_2^2, t_3t_4, t_5t_6, t_4t_6, t_3t_5\right\}\right]^{\frac{1}{2}},$$

where $h \in \left(0, \frac{1}{\sqrt{2}}\right)$. One can show that $F \in \Psi$.

Example 1.7. Define $F : [0, \infty)^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6,$$

where $a > 0$, $b, c, d \geq 0$, $a + b + c < 1$ and $a + d < 1$. One can deduce that $F \in \Psi$.

We state the following theorem for convenience.

Theorem 1.8 (*[12, Theorem 3.1]*). *Let S and I be self-mappings of a metric space (X, d) such that*

(i) *S and I satisfy property* (E. A),
(ii) $\forall x, y \in X$ *and* $F \in \Psi$,

$$F(d(Sx, Sy), d(Ix, Iy), d(Ix, Sx), d(Iy, Sy), d(Ix, Sy), d(Iy, Sx)) \leq 0,$$

(iii) *$I(X)$ is a complete subspace of X.*

Then

(a) *the pair (S, I) has a coincidence point,*
(b) *the pair (S, I) has a common fixed point provided that it is weakly compatible.*

The rest of this paper is organized as follows. In the next section, we introduce a new notion (see, Definition 2.5) which is a generalization of Definition 2.1 [3]. Also, we establish a coincidence and fixed point theorem for two hybrid pairs of nonself-maps satisfying an implicit relation. This theorem generalizes the multivalued version of Theorem 1.8. Also, the same theorem generalizes Theorem 2.8 [4]. Finally, in Section 3, we apply the main theorem for obtaining a coincidence point theorem for hybrid nonself-maps in product spaces.

2. Main results

First we rewrite Definitions 1.1 and 1.2 for nonself-mappings as follows.

Definition 2.1. Let (X, d) be a metric space. Maps $I : Y \subseteq X \to X$ and $S : Y \to CL(X)$ are *weakly compatible* if $Ix \in Sx$ leads to $ISx = SIx$ provided that $Ix \in Y$ and $Sx \subseteq Y$ for all coincidence point $x \in Y$ of I and S.

Definition 2.2. Let (X, d) be a metric space and $S : Y \subseteq X \to CL(X)$. The map $I : Y \to X$ is said to be *S-weakly commuting* at $x \in Y$ if $IIx \in SIx$ provided that $Ix \in Y$ for all $x \in Y$.

Here we remark that the weak compatibility leads to the S-weak commutativity at the coincidence point of I and S but its converse need not be true as the following example.

Example 2.3. Let $X = [1, \infty)$ with the usual metric and $Y = [2, \infty) \subset [1, \infty) = X$. Define $I : Y \to X$ and $S : Y \to CL(X)$ by

$$Ix = x + 2 \quad \text{and} \quad Sx = [2, x + 3],$$

for all $x \in Y$. Then for all $x \in Y$, $Ix \in Sx$,

$$IIx = I(x + 2) = x + 4, \qquad SIx = S(x + 2) = [2, x + 5], \qquad ISx = I[2, x + 3] = [4, x + 5].$$

It clear that $IIx \in SIx$ and $SIx \neq ISx$ for all $x \in Y$. Therefore, I is S-weakly commuting at $x \in Y$ but I and S are not weakly compatible.

Remark 2.4. If S is single-valued nonself-mapping, then S-weak commutativity at the coincidence points is equivalent to the weak compatibility.

Now, we introduce the following definition which is a generalization of Definition 2.1 [3].

Definition 2.5. Let (X, d) be a metric space, and $I, J : Y \subseteq X \to X$ and $T, S : Y \to CL(X)$. The hybrid pair (I, T) is said to be *J-tangential at $t \in Y$ with respect to the map S* if there exist two sequences $\{x_n\}$, $\{y_n\}$ in Y and $A \in CL(X)$ such that $\lim_{n\to\infty} Sy_n \in CL(X)$ and

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A = \lim_{n\to\infty} Tx_n. \tag{1}$$

Remark 2.6. The hybrid pairs (I, T) and (J, S) satisfy the common property (E. A) if and only if (I, T) is J-tangential with respect to S and (J, S) is I-tangential with respect to T. However, the following example shows that if either (I, T) is J-tangential with respect to S or (J, S) is I-tangential with respect to T, then it is not necessary that (I, T) and (J, S) satisfy the common property (E. A).

Example 2.7. Let $X = \Re$ with the usual metric and $Y = [1, \infty)$. Define $I, J : Y \subseteq X \to X$ and $S, T : Y \to CL(X)$ by $Ix = 2 + \frac{1}{2}x$, $Jx = 2 + \frac{1}{3}x$, $Tx = [2, 2 + x]$ and $Sx = [1, 2]$ for all $x \in Y$. Consider the sequences $\{x_n\} = \{1 + \frac{1}{n}\}_{n\in N}$ and $\{y_n\} = \left\{\frac{3}{2} + \frac{1}{n}\right\}_{n\in N}$ in Y where N is the set of all positive integers. Then,

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = \frac{5}{2} \in [2, 3] = \lim_{n\to\infty} Tx_n.$$

Therefore, the hybrid pair (I, T) is J-tangential with respect to S. Further, note that the pair (J, S) is not I-tangential with respect to T. If (I, T) and (J, S) satisfy the common property (E. A), then there exist $\{x_n\}$ and $\{y_n\}$ in Y such that $\lim_{n\to\infty} Tx_n = A$, $\lim_{n\to\infty} Sy_n = B$ and $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A \cap B = \{2\}$. This implies that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. Clearly, there exist no such sequences in Y. Hence, (I, T) and (J, S) do not satisfy the common property (E. A).

Theorem 2.8. *Let I, J be two maps from a subset Y of a metric space (X, d) into X and S, T be two maps from Y into $CL(X)$. Assume that*

(i) *either the pair (I, T) is J-tangential at $t \in Y$ with respect to the map S or the pair (J, S) is I-tangential at $t \in Y$ with respect to the map T,*
(ii) *there exists a function $F \in \Psi$ such that*

$$F(H(Tx, Sy), d(Ix, Jy), d(Ix, Tx), d(Jy, Sy), d(Ix, Sy), d(Jy, Tx)) \leq 0, \tag{2}$$

for all $x, y \in Y$. Then

(a) *I and T have a coincidence point a in Y provided that $I(Y)$ is a closed subset of X;*
(b) *J and S have a coincidence point b in Y provided that $J(Y)$ is a closed subset of X;*

(c) *I and T have a common fixed point provided that I is T-weakly commuting at a, $IIa = Ia$ and $Ia \in Y$;*
(d) *J and S have a common fixed point provided that J is S-weakly commuting at b, $JJb = Jb$ and $Jb \in Y$;*
(e) *I, J, S and T have a common fixed point provided that both (c) and (d) are true.*

Proof. Suppose that (I, T) is J-tangential at $t \in Y$ with respect to S. Then there exist two sequences $\{x_n\}, \{y_n\}$ in Y and $A \in CL(X)$ such that $\lim_{n\to\infty} Sy_n \in CL(X)$ and $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A = \lim_{n\to\infty} Tx_n$. Let $\lim_{n\to\infty} Sy_n = B$. We claim that $A = B$. Suppose not; i.e., $H(A, B) > 0$. Since

$$d(Jy_n, Sy_n) \le d(Jy_n, Tx_n) + H(Tx_n, Sy_n), \qquad d(Ix_n, Sy_n) \le d(Ix_n, Tx_n) + H(Tx_n, Sy_n)$$

and F is non-increasing in the 4th and 5th variables. From (2), we have

$$\begin{aligned}
&F(H(Tx_n, Sy_n), d(Ix_n, Jy_n), d(Ix_n, Tx_n), d(Jy_n, Tx_n) + H(Tx_n, Sy_n), d(Ix_n, Tx_n) + H(Tx_n, Sy_n), d(Jy_n, Tx_n))\\
&\quad \le F(H(Tx_n, Sy_n), d(Ix_n, Jy_n), d(Ix_n, Tx_n), d(Jy_n, Sy_n), d(Ix_n, Sy_n), d(Jy_n, Tx_n))\\
&\quad \le 0.
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$F(H(A, B), 0, 0, H(A, B), H(A, B), 0) \le 0.$$

From the property (ψ_{21}) of $F \in \Psi$, we get that $H(A, B) \le 0$. Thus, $\lim_{n\to\infty} Sy_n = A$.

(a) Suppose that $I(Y)$ is closed, then $\lim_{n\to\infty} Ix_n = t = Ia$ for some $a \in Y$. We claim that $Ta = A$. Suppose not, i.e., $H(Ta, A) > 0$. By (2), we get

$$F(H(Ta, Sy_n), d(Ia, Jy_n), d(Ia, Ta), d(Jy_n, Sy_n), d(Ia, Sy_n), d(Jy_n, Ta)) \le 0.$$

Taking the limit as $n \to \infty$, we have

$$F(H(Ta, A), 0, d(Ta, Ia), 0, 0, d(Ta, Ia)) \le 0. \tag{3}$$

Since $Ia \in A$ and F is non-increasing in the 3rd and 6th variables. Using (3), we obtain

$$F(H(Ta, A), 0, H(Ta, A), 0, 0, H(Ta, A)) \le F(H(Ta, A), 0, d(Ta, Ia), 0, 0, d(Ta, Ia)) \le 0.$$

From the property (ψ_{22}) of $F \in \Psi$, we get that $H(Ta, A) \le 0$. So, $Ta = A$. Therefore, $Ia \in Ta$.

(b) A similar argument gives (b).

(c) By virtue of the conditions in (c), we obtain that $IIa = Ia$ and $IIa \in TIa$. Thus, $t = It \in Tt$ where $t = Ia$.

(d) The proof is similar to the proof of (c).

(e) The result holds immediately.
The proof, assuming that (J, S) is I-tangential at $t \in Y$ with respect to T, is similar to the above. □

The following corollary is a multivalued version of Theorem 1.8 for two single-valued mappings and two multivalued mappings in nonself-arena.

Corollary 2.9. *Let I, J be two maps from a subset Y of a metric space (X, d) into X and S, T be two maps from Y into $CL(X)$ such that*

(i) *T and I satisfy property (E. A) and S and J satisfy property (E. A),*
(ii) *$\forall x, y \in X$ and $F \in \Psi$,*

$$F(H(Tx, Sy), d(Ix, Jy), d(Ix, Tx), d(Jy, Sy), d(Ix, Sy), d(Jy, Tx)) \le 0,$$

(iii) *$I(Y)$ and $J(Y)$ are complete subspaces of X.*

Then

(a) *the pairs (T, I) and (S, J) have a coincidence point,*
(b) *the pairs (T, I) and (S, J) have a common fixed point provided that they are weakly compatible.*

Now, we give an example to show the greater generality of Theorem 2.8 over Corollary 2.9.

Example 2.10. Let $Y = X = [0, \infty)$ endowed with the usual metric. Assume that $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{1}{2}t_2$ for every $t_1, t_2, t_3, t_4, t_5, t_6 \in [0, \infty)$. One can verify that $F \in \Psi$. Define $T, S : X \to CL(X)$ and $I, J : X \to X$ as follows:

$$Tx = [x, \infty), \qquad Sx = [x^2, \infty), \qquad Ix = 2x, \qquad Jx = 2x^2,$$

for all $x \in X$. For all $x, y \in X$, we find that

$$\begin{aligned} H(Tx, Sy) &= |x - y^2| \\ &= \frac{1}{2}|2x - 2y^2| \\ &= \frac{1}{2}d(Ix, Jy). \end{aligned}$$

This gives that

$$F(H(Tx, Sy), d(Ix, Jy), d(Ix, Tx), d(Jy, Sy), d(Ix, Sy), d(Jy, Tx)) = H(Tx, Sy) - \frac{1}{2}d(Ix, Jy) = 0,$$

for all $x, y \in X$. It is clear that $I(X)$ and $J(X)$ are closed subsets of X. One can show that I is T-weakly commuting at 0 and J is S-weakly commuting at 0. Also, $II(0) = I(0)$ and $JJ(0) = J(0)$. But, $JS(1) \neq SJ(1)$ at $J(1) \in S(1)$; i.e., J and S are not weakly compatible. Also, consider the sequences $\{x_n\} = \left\{\frac{1}{n}\right\}_{n \in N}$ and $\{y_n\} = \left\{\frac{1}{2n}\right\}_{n \in N}$ in X. Then,

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = 0 \in [0, \infty) = \lim_{n\to\infty} Tx_n.$$

Therefore, the hybrid pair (I, T) is J-tangential with respect to S. Also, one can deduce that the pair (J, S) is I-tangential with respect to T. We know that 0 is a common fixed point of I, J, T and S. Hence, the hypotheses of Theorem 2.8 are satisfied. Corollary 2.9 is not applicable because J and S are not weakly compatible.

Taking $F : [0, \infty)^6 \to \Re$ as follows

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \lambda \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\},$$

where $\lambda \in (0, 1)$. Then $F \in \Psi$ and from Theorem 2.8, we get the following.

Corollary 2.11. *Let I, J be two maps from a subset Y of a metric space (X, d) into X and S, T be two maps from Y into $CL(X)$. Assume that*

(i) *either the pair (I, T) is J-tangential at $t \in Y$ with respect to the map S or the pair (J, S) is I-tangential at $t \in Y$ with respect to the map T,*
(ii) *there exists $\lambda \in (0, 1)$ such that*

$$H(Tx, Sy) \leq \lambda \max\left\{d(Ix, Jy), d(Ix, Tx), d(Jy, Sy), \frac{d(Ix, Sy) + d(Jy, Tx)}{2}\right\},$$

for all $x, y \in Y$. Then
(a) *I and T have a coincidence point a in Y provided that $I(Y)$ is a closed subset of X;*
(b) *J and S have a coincidence point b in Y provided that $J(Y)$ is a closed subset of X;*
(c) *I and T have a common fixed point provided that I is T-weakly commuting at a, $IIa = Ia$ and $Ia \in Y$;*
(d) *J and S have a common fixed point provided that J is S-weakly commuting at b, $JJb = Jb$ and $Jb \in Y$;*
(e) *I, J, S and T have a common fixed point provided that both* (c) *and* (d) *are true.*

Remark 2.12. Taking $Y = X$ in Corollary 2.11, we obtain a generalization of Theorem 2.8 [4] since

(1) $S(x)$ and $T(x)$ are in $CL(X)$ instead of they are in $CB(X)$ for each $x \in X$;

(2) either the pair (I, T) is J-tangential at $t \in X$ with respect to the map S or the pair (J, S) is I-tangential at $t \in X$ with respect to the map T in lieu of the pairs (I, T) and (J, S) satisfy the common property (E. A).

Corollary 2.13. *Let I, J, S and T be mappings from a subset Y of a metric space (X, d) into X. Assume that*

(i) *either the pair (I, T) is J-tangential at $t \in Y$ with respect to the map S or the pair (J, S) is I-tangential at $t \in Y$ with respect to the map T,*
(ii) *there exists a function $F \in \Psi$ such that*

$$F(d(Tx, Sy), d(Ix, Jy), d(Ix, Tx), d(Jy, Sy), d(Ix, Sy), d(Jy, Tx)) \leq 0, \tag{4}$$

for all $x, y \in Y$. Then
(a) *I and T have a coincidence point a in Y provided that $I(Y)$ is a closed subset of X;*
(b) *J and S have a coincidence point b in Y provided that $J(Y)$ is a closed subset of X;*
(c) *I and T have a common fixed point provided that I is T-weakly commuting at a, $IIa = Ia$ and $Ia \in Y$;*
(d) *J and S have a common fixed point provided that J is S-weakly commuting at b, $JJb = Jb$ and $Jb \in Y$;*
(e) *I, J, S and T have a common fixed point provided that both* (c) *and* (d) *are true.*

Remark 2.14. In fact, in Corollary 2.13, I, J, S and T have a unique common fixed point provided that both (c) and (d) are true.

To see this, let u be another common fixed point of I, J, S and T; i.e., $d(a, u) > 0$.

$$F(d(a,u), d(a,u), 0, 0, d(a,u), d(a,u)) = F(d(Ta, Su), d(Ia, Ju), d(Ia, Ta), d(Ju, Su), d(Ia, Su), d(Ju, Ta)) \leq 0.$$

The last inequality contradicts (ψ_3). Hence, a is the unique common fixed point of I, J, S and T.

Remark 2.15. Suppose that S and T are single-valued mappings. Taking $Y = X$, $S = T$ and $I = J$ in Corollary 2.13, we get a generalization of Theorem 1.8 since

(A) $I(X)$ is a closed subset of X instead of $I(X)$ is a complete subspace of X;

(B) I is T-weakly commuting at a and $IIa = Ia$ for a coincidence point $a \in X$ of T and I in lieu of weak compatibility of the pair (I, T).

3. An application

Motivated by paper [16, Theorem 2], we apply Theorem 2.8 for obtaining a coincidence point theorem for hybrid nonself-maps satisfying an implicit relation in product spaces.

Now, we introduce the following definition.

Definition 3.1. Let Y be a subset of a metric space (X, d). A point $q \in X$ is said to be a *fixed point* of $I : Y \times Y \to X$ (resp. $S : Y \times Y \to CL(X)$) if $q = I(q, u)$ (resp. $q \in S(q, u)$) for all $u \in Y$. The point $q \in X$ is said to be a *common fixed point* of $I : Y \times Y \to X$ and $J : Y \times Y \to X$ (resp. $I : Y \times Y \to X$ and $S : Y \times Y \to CL(X)$) if $q = I(q, u) = J(q, u)$ (resp. $q = I(q, u) \in S(q, u)$) for all $u \in Y$. $q \in X$ is called a *coincidence point* of $I : Y \times Y \to X$ and $J : Y \times Y \to X$ (resp. $I : Y \times Y \to X$ and $S : Y \times Y \to CL(X)$) if $I(q, u) = J(q, u)$ (resp. $I(q, u) \in S(q, u)$) for all $u \in Y$.

Theorem 3.2. *Let Y be a subset of a metric space (X, d), $I, J : Y \times Y \to X$ and $S, T : Y \times Y \to CL(X)$. Assume that*

(i)′ *there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A, B \in CL(X)$ such that*

$$\text{either } \lim_{n\to\infty} S(y_n, u) \in CL(X) \quad \text{and} \quad \lim_{n\to\infty} I(x_n, u) = \lim_{n\to\infty} J(y_n, u) = t \in A = \lim_{n\to\infty} T(x_n, u)$$
$$\text{or } \lim_{n\to\infty} T(x_n, u) \in CL(X) \quad \text{and} \quad \lim_{n\to\infty} I(x_n, u) = \lim_{n\to\infty} J(y_n, u) = t_1 \in B = \lim_{n\to\infty} S(y_n, u)$$

for some $t, t_1 \in Y$ and for every $u \in Y$,

(ii)′ *there exists a function $F \in \Psi$ such that*

$$F(H(S(x,u), T(y,u')), d(I(x,u), J(y,u')), d(I(x,u), S(x,u)),$$
$$d(J(y,u'), T(y,u')), d(I(x,u), T(y,u')), d(J(y,u'), S(x,u))) \leq 0, \tag{5}$$

for all $x, y, u, u' \in Y$. Then

(a)′ *If $I(Y \times Y)$ is a closed subset of X, then there exists $b \in Y$ such that $I(b, u) \in S(b, u)$ for all $u \in Y$;*

(b)′ *If $J(Y \times Y)$ is a closed subset of X, then there exists $c \in Y$ such that $J(c, u') \in T(c, u')$ for all $u' \in Y$.*

Proof. By (5), we find that

$$F(H(S(x,u), T(y,u)), d(I(x,u), J(y,u)), d(I(x,u), S(x,u)),$$
$$d(J(y,u), T(y,u)), d(I(x,u), T(y,u)), d(J(y,u), S(x,u))) \leq 0,$$

for all $x, y, u \in Y$. From Theorem 2.8(a), there exists $b \in Y$ such that $I(b, u) \in S(b, u)$ for all $u \in Y$. Similarly, one can prove (b)′. □

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References

[1] T. Kamran, Coincidence and fixed points for hybrid strict contractions, J. Math. Anal. Appl. 299 (2004) 235–241.
[2] T. Kamran, Fixed points of asymptotically regular noncompatible maps, Demonstratio Math. 38 (2) (2005) 485–494.

[3] T. Kamran, N. Cakić, Hybrid tangential property and coincidence point theorems, Fixed Point Theory 9 (2) (2008) 487–496.
[4] Y. Liu, J. Wu, Z. Li, Common fixed points of single-valued and multivalued maps, Int. J. Math. Math. Sci. 19 (2005) 3045–3055.
[5] S.A. Naimpally, S.L. Singh, J.H.M. Whitfield, Coincidence theorems for hybrid contractions, Math. Nachr. 127 (1986) 177–180.
[6] S.L. Singh, S.N. Mishra, Coincidence and fixed points of nonself hybrid contractions, J. Math. Anal. Appl. 256 (2) (2001) 486–497.
[7] H.K. Pathak, S.M. Kang, Y.J. Cho, Coincidence and fixed point theorems for nonlinear hybrid generalized contractions, Czechoslovak Math. J. 48 (123) (1998) 341–357.
[8] M.A. Ahmed, B.E. Rhoades, Some common fixed point theorems for compatible mappings, Indian J. Pure Appl. Math. 32 (8) (2001) 1247–1254.
[9] Lj.B. Ciric, Contractive type non-self mappings on metric spaces of hyperbolic type, J. Math. Anal. Appl. 317 (2006) 28–42.
[10] Lj.B. Ciric, J.S. Ume, M.S. Khan, H.K. Pathak, On some nonself mappings, Math. Nachr. 251 (2003) 28–33.
[11] T. Kamran, Coincidence and fixed points of contractive type multivalued maps, Georgian Math. J. 15 (1) (2008) 36–70.
[12] M. Imdad, J. Ali, Jungck's common fixed point theorem and EA property, Acta Math. Sinica 25 (2007) 1–8.
[13] V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, Demonstratio Math. 32 (1) (1999) 157–163.
[14] G. Jungck, B.E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 16 (3) (1998) 227–238.
[15] M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181–188.
[16] S.L. Singh, B.M.L. Tiwari, V.K. Gupta, Common fixed points of commuting mappings in 2-metric spaces and an application, Math. Nachr. 95 (1980) 293–297.